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NET-THEORETICAL L-GENERALIZED CONVERGENCE SPACES

W. YAO

Abstract. In this paper, the definition of net-theoretical L-generalized convergence spaces is proposed. It is shown that, for L a frame, the category of enriched L-fuzzy topological spaces can be embedded in that of L-generalized convergence spaces as a reflective subcategory and the latter is a cartesian-closed topological category.

1. **Introduction**

It is well-known that the category **Top** of topological spaces and continuous maps fails to have some desirable properties, for example, there is in general no natural function space topology, that is, **Top** is not cartesian-closed (see in [11] for a detailed proof). Because of this fact, **Top** is inconvenient for investigations in algebraic topology (homotopy topology), functional analysis (duality theory) or topological algebra (quotients). It is natural for us to look for some well-behaved subcategories or more convenient supercategories.

Convergence theory of filters or nets provides a good tool for interpreting topological structures and plays an important role in topology. In the crisp situation, there are close relations between topological spaces and convergence spaces. For a nonempty set X, let $\mathbb{F}(X)$ denote the set of all filters (which are equivalent to proper lattice-theoretical filters of $(2^X,\subseteq)$) on X and for each $x\in X$, let $[x]$ denote the principal filter generated by $\{x\}$. A *filter-theoretical convergence structure* (or called a *convergence relation of filters* [8]) on X is a subset $R\subseteq\mathbb{F}(X)\times X$ satisfying the following conditions:

(FGC1) $([x],x)\in R$ for all $x\in X$;
(FGC2) $(F,x)\in R$ and $F\subseteq G\in\mathbb{F}(X)$ imply $(G,x)\in R$.

The pair (X,R) is called a *generalized convergence space.* A map $f:(X,R_X)\longrightarrow(Y,R_Y)$ between two generalized convergence spaces is called *continuous* if for all $\mathcal{F}\in\mathbb{F}(X)$, $(\mathcal{F},x)\in R_X$ implies $(f(\mathcal{F}),f(x))\in R_Y$, where $f(\mathcal{F})$ is a filter on Y generated by the filter base $\{f(A)|\ A\in\mathcal{F}\}$. The category of filter-theoretical generalized convergence spaces with continuous maps as morphisms is denoted by **FGConv**. It is well-known that the category of topological spaces **Top** can be

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embedded in **FGConv** as a reflective subcategory, and the latter is cartesian-closed [11].

Dually, generalized convergence spaces can also be defined by means of nets. Let $\mathbb{N}(X)$ denote the set of all nets on X and for each $x \in X$, let $\overline{x}$ denote the constant net at x. A *net-theoretical convergence structure* (or called a *convergence relation of nets*) on X is a subset $S \subseteq \mathbb{N}(X) \times X$ satisfying the following conditions:

(NGC1) $(\overline{x}, x) \in S$ for all $x \in X$;
(NGC2) If $(\xi, x) \in S$ and η is subnet of ξ, then $(\eta, x) \in S$.

The pair (X, S) is called a *net-theoretical generalized convergence space*.

In fuzzy setting, there are a lot of papers dealing with convergence. Lowen [10] gave a definition of convergence for stratified [0,1]-topological spaces, where prime prefilters play a crucial role. In [2], Höhle pointed out that this theory may turn out to be void in case of more general lattices, and further he suggested to develop a theory and analogous convergence theory based on the concept of L-filters [4]. Following this suggestion, Jäger [5] developed a theory of convergence based on the notion of L-filter in case of L being a complete Heyting algebra. It is proved that **SL-GCS**, the category of stratified L-generalized convergence spaces (which are called stratified L-fuzzy convergence spaces in [5]), is a cartesian-closed topological category and contains the category of stratified L-topological spaces as a reflective subcategory. For the case of nets, by defining fuzzy nets and their convergence to fuzzy points, Pu and Liu [12] established the More-Smith convergence theory in fuzzy topology. In [9], it was pointed out that L-convergence classes and L-topologies completely determines each other. In [16], Yue and Fang defined a kind of convergence on completely distributive lattices in sense of pointless topology.

In [15], we define and study an L-generalized convergence space using L-filters of ordinary sets. The aim of this paper is to introduce the concept of net-theoretical L-generalized convergence spaces using ordinary nets and to show the resulting category is a cartesian-closed topological category and the category of enriched L-fuzzy topological spaces can be embedded in it as a reflective subcategory.

2. Preliminaries

In this section, we shall recall some basic definitions and results which will be used in the following sections.

2.1. **Lattices and Fuzzy Sets.** Let L be a complete lattice. If the binary meets are distributive over arbitrary joins, that is, for all $a \in L$ and all $B \subseteq L$, $a \wedge \bigvee B = \bigvee_{b \in B}(a \wedge b)$, then L is called a *frame* or a *complete Heyting algebra* [7]. For a frame L, we can define a binary operation $\rightarrow: L \times L \longrightarrow L$ by $a \rightarrow b = \bigvee\{c \in L \mid c \wedge a \leq b\}$ $(\forall a, b \in L)$. We have $a \wedge b \leq c \Longleftrightarrow a \leq b \rightarrow c$ $(\forall a, b, c \in L)$. Throughout this paper, L always denotes a frame. Properties of frames can be found in many papers, for example [4].

For a set X, L^X denotes the set of all maps from X to L and every member of L^X is called an L-subset of X. The notations $0_X, 1_X$ denote the constant L-subsets of X with the value $0, 1$ respectively. For $\lambda \in L$ we denote the constant L-subset with value λ again by λ.

Let $f : X \longrightarrow Y$ be an ordinary map, define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called L-valued Zadeh function or L-forward powerset operator, cf. [13]) and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ (called L-backward powerset operator, cf. [13]) by $f_L^{\rightarrow}(a)(y) = \bigvee_{f(x)=y} a(x)$ for $a \in L^X$ and $y \in Y$, and $f_L^{\leftarrow}(b) = b \circ f$ for $b \in L^Y$, respectively.

2.2. **Related Category Theory.** A concrete category $\mathcal{C}$ over **Set** (U is the forgetful functor) is called *topological* if for any U-source $(f_i \longrightarrow (X_i, \xi_i))_{i \in I}$, there exists a unique initial U-lift (X, ξ), that is for any $\mathcal{C}$-object (Y, η), a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a $\mathcal{C}$-morphism if and only if for any $i \in I$ $f_i \circ g : (Y, \eta) \longrightarrow (X_i, \xi_i)$ is a $\mathcal{C}$-morphism.

A pair of functors (F, G) is called an *adjunction* between two categories $\mathcal{A}$ and $\mathcal{B}$ if for any $A \in ob(\mathcal{A}), B \in ob(\mathcal{B})$, there is a bijection between $hom_{\mathcal{A}}(A, G(B))$ and $hom_{\mathcal{B}}(F(A), B)$. The functor F is called the *left adjoint* of G and G the *right adjoint* of F. If $\mathcal{A}$ is a subcategory of $\mathcal{B}$ and the inclusion functor $i : \mathcal{A} \longrightarrow \mathcal{B}$ has a left (resp., right) adjoint, then $\mathcal{A}$ is called a *reflective* (resp., *coreflective*) subcategory of $\mathcal{B}$.

A category with finite products is called *cartesian-closed* if for each pair (A, B) of objects there exists an object $[A \rightarrow B]$ and an evaluation morphism $ev : [A \rightarrow B] \times A \longrightarrow B$ with the following universal property: for each morphism $f : C \times A \longrightarrow B$ there exists a unique morphism $\hat{f} : C \longrightarrow [A \rightarrow B]$ such that $ev \circ (\hat{f} \times id_A) = f$.

We refer to [1] for other notions and properties related to category theory.

2.3. **L-topology and L-fuzzy Topology.** An *L-fuzzy topology* [4] on a nonempty set X is a function $\tau : L^X \longrightarrow L$ which satisfies: (O1) $\tau(0_X) = \tau(1_X) = 1$; (O2) $\tau(a \wedge b) \geq \tau(a) \wedge \tau(b)$; (O3) $\tau(\bigvee_i a_i) \geq \bigwedge_i \tau(a_i)$. Further, τ is called *enriched* if (E) $\tau(\lambda) = 1$ for all $\lambda \in L$, or (E') $\tau(\lambda \wedge a) \geq \tau(a)$ for all $\lambda \in L, a \in L^X$. For an (enriched) L-fuzzy topology τ on X, the pair (X, τ) is called an *(enriched) L-fuzzy topological space.*

Let (X, τ_1) and (Y, τ_2) be two L-fuzzy topological spaces. A map $f : X \longrightarrow Y$ is called *continuous* if $\tau_1(f_L^{\leftarrow}(b)) \geq \tau_2(b)$ holds for all $b \in L^Y$. The category of L-fuzzy topological spaces with L-fuzzy continuous maps as morphisms will be denoted by L-**FTop**. We use **R**L-**FTop** denote the full subcategory of L-**FTop** formed by all enriched L-fuzzy topological spaces.

2.4. **Nets and (Stratified) L-filters.** Let X be a nonempty set. A *net* ξ of X is a map $\xi : \Delta \longrightarrow X$, where Δ is a directed set, denoted by $\xi = (x_\delta)_{\delta \in \Delta}$ or just $(x_\delta)_{\delta \in \Delta}$. Let $\xi : D \longrightarrow X$ and $\eta : E \longrightarrow X$ be two nets. A net η is called a *subnet* of ξ if there is a map $h : E \longrightarrow D$ that satisfies $\eta = \xi \circ h$ and for each $d \in D$, there exists an $e_0 \in E$ such that $h(e) \geq d$ for all $e \geq e_0$.

A constant net valued at x is denoted by $\overline{x}$. Let $\xi : D \longrightarrow X$ be a net and $f : X \longrightarrow Y$. Then $f(\xi) = f \circ \xi : D \longrightarrow Y$ is a net, called the *image of* ξ *under* f. Let $\xi : D \longrightarrow X$, $\eta : E \longrightarrow Y$ be two nets. Then $\xi \times \eta : D \times E \longrightarrow X \times Y$ by $(\xi \times \eta)(d, e) = (\xi(d), \eta(e))$ is a net of $X \times Y$.

A map $\mathcal{F} : L^X \longrightarrow L$ is called an *L-filter* [4, 5] on a nonempty set X if it satisfies (F1) $\mathcal{F}(0_X) = 0, \mathcal{F}(1_X) = 1$; (F2) $\mathcal{F}(a \wedge b) = \mathcal{F}(a) \wedge \mathcal{F}(b)$ for all $a, b \in L^X$. An L-filter $\mathcal{F}$ is called *stratified* if (Fs) $\mathcal{F}(\lambda) \geq \lambda$ for all $\lambda \in L$, or (Fs$'$) $\lambda \wedge \mathcal{F}(a) \leq \mathcal{F}(\lambda \wedge a)$ for all $\lambda \in L, a \in L^X$.

Define $[x] : L^X \longrightarrow L$ by $[x](a) = a(x)$. Then $[x]$ is a stratified L-filter, called the *pointed stratified L-filter* of x [4]. Let (X, τ) be an L-fuzzy topological space. For each $x \in X$, define $\mathcal{U}_\tau^x : L^X \longrightarrow L$ by $\mathcal{U}_\tau^x(a) = \bigvee_{b \leq a} b(x) \wedge \tau(b)$ $(\forall a \in L^X)$. Then $\mathcal{U}_\tau^x$ is an L-filter [14] and if τ is enriched then $\mathcal{U}_\tau^x$ is stratified.

Let $\mathcal{F}$ be an L-filter on X and $f : X \longrightarrow Y$ be a map. Define $f(\mathcal{F}) : L^Y \longrightarrow L$ by $f(\mathcal{F})(b) = \mathcal{F}(f_L^{\leftarrow}(b))$ $(\forall b \in L^Y)$. Then $f(\mathcal{F})$ is an L-filter on Y and if $\mathcal{F}$ is stratified then so is $f(\mathcal{F})$ [4]. If F is an ordinary filter on X, then $f(F) = \{B \subseteq Y |\ f^{-1}(B) \in F\}$.

For a net $\xi : D \longrightarrow X$, define $\mathcal{F}_\xi : L^X \longrightarrow L$ by

$$\mathcal{F}_\xi(a) = \bigvee_{d_0 \in D} \bigwedge_{d \geq d_0} a(\xi(d))\ (\forall a \in L^X).$$

In [3], for an ordinary filter F on X, an L-filter $\mathcal{F}_F$ can be defined by

$$\mathcal{F}_F(a) = \bigvee_{A \in F} \bigwedge_{x \in A} a(x).$$

Thus the construction of $\mathcal{F}_\xi$ can be considered as a special case of $\mathcal{F}_F$ by taking F the associated filter of ξ.

Lemma 2.1. (1) $\mathcal{F}_F$ is a stratified L-filter on X.

(2) If $F \leq G$, then $\mathcal{F}_F \leq \mathcal{F}_G$.

(3) For all $x \in X$, $\mathcal{F}_{\dot{x}} = [x]$, where $\dot{x}$ is the pointed (ordinary) filter.

(4) $\mathcal{F}_{f(F)} = f(\mathcal{F}_F)$.

Proof. (1), (2) and (3) are trivial and straightforward.

(4) For all $b \in L^Y$, $f(\mathcal{F}_F)(b) = \mathcal{F}_F(f_L^{\leftarrow}(b))$,

$$\mathcal{F}_{f(F)}(b) = \bigvee_{B \in f(F)} \bigwedge_{y \in B} b(y) \leq \bigvee_{f^{-1}(B) \in F} \bigwedge_{f(x) \in B} b(f(x)) \leq \bigvee_{A \in F} \bigwedge_{x \in A} f_L^{\leftarrow}(b)(x) = \mathcal{F}_F(f_L^{\leftarrow}(b))$$

and

$$\mathcal{F}_F(f_L^{\leftarrow}(b)) = \bigvee_{A \in F} \bigwedge_{x \in A} f_L^{\leftarrow}(b)(x) \leq \bigvee_{f(A) \in f(F)} \bigwedge_{x \in A} b(f(x)) \leq \bigvee_{B \in f(F)} \bigwedge_{y \in B} b(y) = \mathcal{F}_{f(F)}(b).$$

Hence $\mathcal{F}_{f(F)} = f(\mathcal{F}_F)$. □

Proposition 2.2. (1) $\mathcal{F}_\xi$ is a stratified L-filter on X.

(2) If η is a subnet of ξ, then $\mathcal{F}_\xi \leq \mathcal{F}_\eta$.

(3) For all $x \in X$, $\mathcal{F}_{\overline{x}} = [x]$.

(4) $\mathcal{F}_{f(\xi)} = f(\mathcal{F}_\xi)$.

Proof. (1–4) hold since it is straightforward to verify that (a) $\mathcal{F}_\xi = \mathcal{F}_{F_\xi}$, where F_ξ is the associated filter with respect to ξ; (b) if $F \leq G$ then ξ_G is a subnet of ξ_F; (c) $F_{\overline{x}} = \dot{x}$; (d) $F_{f(\xi)} = f(F_\xi)$. □

3. Net-theoretical L-generalized Convergence Spaces and Their Relations with L-fuzzy Topological Spaces

In this section, we will define an L-generalized convergence spaces using ordinary nets and then show the resulting category can embed the category of enriched L-fuzzy topological spaces as a reflective subcategory.

Definition 3.1. A map $S : \mathbb{N}(X) \times X \longrightarrow L$ is called an *L-generalized convergence structure* on X if

(NGC1) For all $x \in X,\ S(\overline{x}, x) = 1$;

(NGC2) If η is a subnet of ξ, then for any $x \in X,\ S(\xi, x) \leq S(\eta, x)$.

The pair (X, S) is called an *L-generalized convergence space*.

For two L-generalized convergence spaces (X, S_1) and (Y, S_2), a map $f : X \longrightarrow Y$ is called *continuous* if for any $(\xi,\ x) \in \mathbb{N}(X) \times X,\ S_1(\xi, x) \leq S_2(f(\xi), f(x))$. Denote by L-**NGCS** the category of L-generalized convergence spaces with continuous maps as morphisms.

Let (X, S) be an L-generalized convergence space. Define $\mathcal{U}_S^x : L^X \longrightarrow L$ by

$$\mathcal{U}_S^x(a) = \bigwedge_{\xi \in \mathbb{N}(X)} S(\xi, x) \to \mathcal{F}_\xi(a)\ (\forall a \in L^X)$$

and define $\tau_S : L^X \longrightarrow L$ by

$$\tau_S(a) = \bigwedge_{x \in X} a(x) \to \mathcal{U}_S^x(a)\ (\forall a \in L^X).$$

Notice that τ_S is the same as $o_{\lim}$ in [5] (cf. Proposition 7.1) with a different neighborhood L-filter. It is routine to verify the following lemma.

Lemma 3.2. (1) For all $x \in X$, $\mathcal{U}_S^x$ is a stratified L-filter.

(2) Let $f : (X, S_1) \longrightarrow (Y, S_2)$ be a continuous map. Then $f_L^{\rightarrow}(\mathcal{U}_{S_1}^x) \geq \mathcal{U}_{S_2}^{f(x)}$.

Proof. (1) is straightforward.

(2) For all $b \in L^Y$,

$$\begin{aligned}
& f_L^{\rightarrow}(\mathcal{U}_{S_1}^x)(b) = \mathcal{U}_{S_1}^x(f_L^{\leftarrow}(b)) \\
= & \bigwedge_{\xi \in \mathbb{N}(X)} S_1(\xi, x) \to \mathcal{F}_\xi(f_L^{\leftarrow}(b)) \\
\geq & \bigwedge_{\xi \in \mathbb{N}(X)} S_2(f(\xi), f(x)) \to f(\mathcal{F}_\xi)(b) \\
= & \bigwedge_{\xi \in \mathbb{N}(X)} S_2(f(\xi), f(x)) \to \mathcal{F}_{f(\xi)}(b) \\
\geq & \bigwedge_{\eta \in \mathbb{N}(Y)} S_2(\eta, f(x)) \to \mathcal{F}_\eta(b) \\
= & \mathcal{U}_{S_2}^{f(x)}(b).
\end{aligned}$$

□

Proposition 3.3. The map τ_S is an enriched L-fuzzy topology on X.

Proof. (O1) Obviously, $\tau_S(0_X)=\tau_S(1_X)=1$.

(O2)

$$\begin{aligned} & \tau_S(a)\wedge\tau_S(b)\\ = & \bigwedge_{x\in X} a(x)\to\mathcal{U}_S^x(a)\wedge\bigwedge_{y\in X} b(y)\to\mathcal{U}_S^y(b)\\ \le & \bigwedge_{x\in X}(a(x)\to\mathcal{U}_S^x(a))\wedge(b(x)\wedge\mathcal{U}_S^x(b))\\ \le & \bigwedge_{x\in X}(a(x)\wedge b(x))\to(\mathcal{U}_S^x(a)\wedge\mathcal{U}_S^x(b))\\ = & \bigwedge_{x\in X}(a\wedge b)(x)\to\mathcal{U}_S^x(a\wedge b)\\ = & \tau_S(a\wedge b). \end{aligned}$$

(O3)

$$\begin{aligned} \tau_S(\textstyle\bigvee_i a_i) &= \bigwedge_{x\in X}(\textstyle\bigvee_i a_i)(x)\to\mathcal{U}_S^x(\textstyle\bigvee_i a_i)\\ &= \textstyle\bigwedge_i(\displaystyle\bigwedge_{x\in X} a_i(x)\to\mathcal{U}_S^x(\textstyle\bigvee_j a_j))\\ &\ge \textstyle\bigwedge_i(\displaystyle\bigwedge_{x\in X} a_i(x)\to\mathcal{U}_S^x(a_i))\\ &= \textstyle\bigwedge_i \tau_S(a_i). \end{aligned}$$

(E) $\tau_S(\lambda)=\bigwedge\limits_{x\in X}\lambda\to\mathcal{U}_S^x(\lambda)=1$. □

Proposition 3.4. If $f:(X,S_1)\longrightarrow(Y,S_2)$ is continuous, then $f:(X,\tau_{S_1})\longrightarrow(Y,\tau_{S_2})$ is continuous.

Proof. For all $b\in L^Y$,

$$\begin{aligned} \tau_{S_1}(f_L^{\leftarrow}(b)) &= \bigwedge_{x\in X} f_L^{\leftarrow}(b)(x)\to\mathcal{U}_{S_1}^x(f_L^{\leftarrow}(b))\\ &= \bigwedge_{x\in X} b(f(x))\to f_L^{\rightarrow}(\mathcal{U}_{S_1}^x)(b)\\ &= \bigwedge_{x\in X} b(f(x))\to\mathcal{U}_{S_2}^{f(x)}(b)\\ &\ge \tau_{S_2}(b). \end{aligned}$$

□

By Propositions 3.3 and 3.4, we obtain a concrete functor T_S from **EL-FTop** to L-**NGCS** transferring an L-generalized convergence structure S to τ_S.

Let (X,τ) be an enriched L-fuzzy topological space. Define $S_\tau:\mathbb{N}(X)\times X\longrightarrow L$ by

$$S_\tau(\xi,x)=R_\tau(\mathcal{F}_\xi,x)=\bigwedge_{a\in L^X}(\tau(a)\wedge a(x))\to\mathcal{F}_\xi(a)\ (\forall(\xi,x)\in\mathbb{N}(X)\times X).$$

Proposition 3.5. For an enriched L-fuzzy topology τ on X, we have

(1)S_τ is an L-generalized convergence structure on X.

(2) For each $x\in X$, $\mathcal{U}_{S_\tau}^x\ge\mathcal{U}_\tau^x$.

Proof. (1) is trivial by Proposition 2.2.

(2) For all $a \in L^X$ and all $b \leq a$,

$$\begin{aligned}
\mathcal{U}^x_{S_\tau}(a) &\geq \mathcal{U}^x_{S_\tau}(b) = \bigwedge_{\xi\in\mathbb{N}(X)} S_\tau(\xi,x) \to \mathcal{F}_\xi(b)\\
&= \bigwedge_{\xi\in\mathbb{N}(X)} \Big(\bigwedge_{c\in L^X} (\tau(c)\wedge c(x)) \to \mathcal{F}_\xi(c)\Big) \to \mathcal{F}_\xi(b)\\
&\geq \bigwedge_{\xi\in\mathbb{N}(X)} ((\tau(b)\wedge b(x)) \to \mathcal{F}_\xi(b)) \to \mathcal{F}_\xi(b)\\
&\geq \tau(b)\wedge b(x),
\end{aligned}$$

then $\mathcal{U}^x_{S_\tau} \geq \mathcal{U}^x_\tau$. □

Proposition 3.6. If $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ is continuous, then so is $f : (X, S_{\tau_1}) \longrightarrow (Y, S_{\tau_2})$.

Proof. For all $(\xi, x) \in \mathbb{N}(X) \times X$,

$$\begin{aligned}
S_{\tau_2}(f(\xi), f(x)) &= \bigwedge_{b\in L^Y} (\tau_2(b)\wedge b(f(x))) \to \mathcal{F}_{f(\xi)}(b)\\
&\geq \bigwedge_{b\in L^Y} (\tau_1(f^{\leftarrow}_L(b))\wedge f^{\leftarrow}_L(b)(x)) \to \mathcal{F}_\xi(f^{\leftarrow}_L(b))\\
&\geq \bigwedge_{A\in L^X} (\tau_1(a)\wedge a(x)) \to \mathcal{F}_\xi(a)\\
&= S_{\tau_1}(\xi, x).
\end{aligned}$$

□

By Propositions 3.5 and 3.6, we obtain a concrete functor S_T from **RL-FTop** to **L-NGCS** transferring an enriched L-fuzzy topology τ to S_τ.

Theorem 3.7. Let τ be an enriched L-fuzzy topology and S be a net-theoretical L-generalized convergence space. Then
(1) $S_{\tau_S} \geq S$;
(2) $\tau_{S_\tau} \geq \tau$.

Proof. (1) For all $(\xi, x) \in \mathbb{N}(X) \times X$,

$$\begin{aligned}
S_{\tau_S}(\xi,x) &= \bigwedge_{a\in L^X} (\tau_S(a)\wedge a(x)) \to \mathcal{F}_\xi(a)\\
&\geq \bigwedge_{a\in L^X} ((S(\xi,x)\wedge a(x)) \to \mathcal{F}_\xi(a))\wedge a(x)) \to \mathcal{F}_\xi(a)\\
&\geq \bigwedge_{a\in L^X} (S(\xi,x) \to \mathcal{F}_\xi(a)) \to \mathcal{F}_\xi(a)\\
&\geq S(\xi,x).
\end{aligned}$$

(2) For all $a \in L^X$,

$$\begin{aligned}
\tau_{S_\tau}(a) &= \bigwedge_{x\in X} a(x) \to \mathcal{U}_{S_\tau}(a)\\
&\geq \bigwedge_{x\in X} a(x) \to \mathcal{U}_\tau(a)\\
&\geq \bigwedge_{x\in X} a(x) \to (a(x)\wedge\tau(a))\\
&\geq \tau(a).
\end{aligned}$$

□

Theorem 3.8. The category **EL-FTop** can be embedded in **L-NGCS** as a reflective subcategory.

Proof. By Theorem 3.7, we know that (T_S, S_T) is an adjoint between **E**L-**FTop** and L-**NGCS**. Hence, **E**L-**FTop** can be embedded in L-**NGCS** as a reflective subcategory. □

4. L-NGCS is a Cartesian-closed Topological Category

In this section, we will show that L-**NGCS** is a cartesian-closed topological category. The approach in this section is to translating the filter-theoretical results in [5] to net-theoretical ones.

Theorem 4.1. The category L-**NGCS** is topological over **Set**.

Proof. Let $U : L$-**NGCS** $\longrightarrow$ **Set** be the forgetful functor and $(X, f_i,\ (X_i, S_i))_{i\in I}$ be a U-source. Define $S : \mathbb{N}(X) \times X \longrightarrow L$ by

$$S(\xi, x) = \bigwedge_i S_i(f_i(\xi), f_i(x)).$$

Then (X, S) is the initial lift of the given source. □

Let $\{(X_i, S_i)|\ i \in I\}$ be a nonempty family of L-generalized convergence spaces and $X = \prod\limits_{i\in X} X_i$. For $(\xi, x) \in \mathbb{N}(X) \times X$, let $S(\xi, x) = \bigwedge\limits_{i\in I} S_i(p_i(\xi), p_i(x))$. It is easy and straightforward to verify that (X, S) is the product of $\{(X_i, S_i)|\ i \in I\}$ in L-**NGCS**.

Let (X, S_X) and (Y, S_Y) be two L-generalized convergence space and $[X \to Y]$ the set of all continuous maps from (X, S_X) to (Y, S_Y). For all $(\xi, f) \in \mathbb{N}([X \to Y]) \times [X \to Y]$, define

$$S_{[X\to Y]}(\xi, f) = \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(ev(\xi \times \eta), f(x))).$$

Lemma 4.2. For $f \in [X \to Y]$ and $\eta \in \mathbb{N}(X)$, $ev(\overline{f} \times \eta)$ is a subnet of $f(\eta)$, where $\overline{f}$ is the constant net on $[X \to Y]$.

Proof. Let $\overline{f} : D \longrightarrow [X \to Y]$, $\eta : E \longrightarrow X$ be two nets. Then the net $ev(\overline{f} \times \eta) : D\times E \longrightarrow Y$ is defined by $(d, e) \mapsto f(\eta(e))$ and that $f(\eta) : E \longrightarrow Y$ by $e \mapsto f(\eta(e))$. Define $h : D \times E \longrightarrow E$ by $(d, e) \mapsto e$. Then it is straightforward to verify that $ev(\overline{f} \times \eta)$ is a subnet of $f(\eta)$. □

Lemma 4.3. Let ξ_1 be a subnet ξ_2 on X. Then for any net η on Y and any map $f : X \times Y \longrightarrow Z$, $f(\xi_1 \times \eta)$ is a subnet of $f(\xi_2 \times \eta)$.

Proof. Straightforward. □

Proposition 4.4. Let (X, S_X) and (Y, S_Y) be two L-generalized convergence space. Then $S_{[X\to Y]}$ is an L-generalized convergence structure on $[X \to Y]$.

Proof. (NGC2) can be easily implied by Lemma 4.2. (NGC1) By Lemma 4.1,

$$\begin{aligned} S_{[X\to Y]}(\overline{f}, f) &= \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(ev(\overline{f}\times\eta), f(x))) \\ &\geq \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(f(\eta), f(x))) \\ &= 1. \end{aligned}$$

□

Proposition 4.5. The evaluation $ev : ([X \to Y],\ S_{[X\to Y]}) \times (X, R_X) \longrightarrow (Y, R_Y)$ is a continuous map.

Proof. For all $(\xi, f) \in \mathbb{N}([X \to Y]) \times [X \to Y]$ and all $(\eta, x) \in \mathbb{N}(X) \times X$,

$$\begin{aligned} S_{[X\to Y]}(\xi, f) &= \bigwedge_{(\beta,a)\in\mathbb{N}(X)\times X} S_X(\beta, a) \to S_Y(ev(\xi\times\beta), f(a))) \\ &\leq S_X(\eta, x) \to S_Y(ev(\xi\times\eta), f(x))) \end{aligned}$$

Then

$$S_{[X\to Y]\times X}((\xi,\eta),\ (f,x)) = S_{[X\to Y]}(\xi, f) \wedge S_X(\eta, x) \leq S_Y(ev(\xi\times\eta),\ ev(f,x)).$$

That is ev is continuous. □

Now let us consider the following situation. Let $f : X \times Y \longrightarrow Z$ be a map. Define for $x \in X$ the map $f_x : Y \longrightarrow Z,\ y \longmapsto f(x,y)$ and with this the map $f^* : X \longrightarrow Z^Y,\ x \longmapsto f_x$. The map $\varphi : Z^{X\times Y} \longrightarrow (Z^Y)^X,\ f \longmapsto f^*$ is called the exponential map.

Lemma 4.6. Let $f : X \times Y \longrightarrow Z$ be a map and $\xi : D \longrightarrow Y$ a net. Suppose that $\overline{x} : D \longrightarrow X$ be the constant net. Then $f_x(\xi)$ is a subnet of $f(\overline{x} \times \xi)$.

Proof. The net $f(\overline{x}\times\xi) : D\times D \longrightarrow Z$ is defined by $f(\overline{x}\times\xi)(e,\ d) = f(x, \xi(d))$ and the net $f_x(\xi) : D \longrightarrow Z$ is defined by $f_x(\xi)(d) = f(x, \xi(d))$. Define $h : D \longrightarrow D\times D$ by $h(d) = (d, d)$. Then we have $f_x(\xi) = f(\overline{x}\times\xi)\circ h$. It is straightforward to verify that $f_x(\xi)$ is a subnet of $f(\overline{x}\times\xi)$. □

Lemma 4.7. Let $f : (X, S_X) \times (Y, S_Y) \longrightarrow (Z, S_Z)$ be continuous. Then for each $x \in X$ also $f_x : (Y, S_Y) \longrightarrow (Z, S_Z)$ is continuous.

Proof. $\forall(\xi, y) \in \mathbb{N}(Y) \times Y$,

$$S_Z(f_x(\xi), f_x(y)) \geq S_Z(f(\overline{x}\times\xi), f(x,y))$$

$$\geq S_{X\times Y}(\overline{f}\times\xi, (x,y)) = S_X(\overline{x}, x) \wedge S_Y(\xi, y) = S_Y(\xi,\ y).$$

Thus f_x is continuous. □

Lemma 4.8. For all $\xi \in \mathbb{N}(X)$ and all $\eta \in \mathbb{N}(Y)$, $f : X \times Y \longrightarrow Z$, we have $ev(\varphi(f)(\xi)\times\eta) = f(\xi\times\eta)$.

Proof. Let $\xi : D \longrightarrow X$ and $\eta : E \longrightarrow Y$ be two nets. $\varphi(f)(\xi) : D \longrightarrow Z^Y$ is defined by $d \mapsto f_{\xi(d)}$. And $ev(\varphi(f)(\xi)\times\eta) : D\times E \longrightarrow Z$ is defined by $(d,e) \mapsto ev(f_{\xi(d)}, \eta(e)) = f(\xi(d), \eta(e))$. Therefore, $ev(\varphi(f)(\xi)\times\eta) = f(\xi\times\eta)$. □

Proposition 4.9. If the map $f : X \times Y \longrightarrow Z$ is continuous, then so is $\varphi(f) : X \longrightarrow [Y \to Z]$.

Proof. φ is well-defined. For $(\xi,\ x) \in \mathbb{N}(X) \times X$,

$$S_{[Y\to Z]}(\varphi(f)(\xi), \varphi(f)(x)) = \bigwedge_{(\eta,y)\in\mathbb{N}(Y)\times Y} S_Y(\eta, y) \to S_Z(ev(\varphi(f)(\xi) \times \eta), f_x(y))).$$

For $(\eta, y) \in \mathbb{N}(Y) \times Y$,

$$\begin{aligned} S_Y(\eta, y) \to S_Z(ev(\varphi(f)(\xi) \times \eta), f_x(y))) &= S_Y(\eta, y) \to S_Z(f(\xi \times \eta), f(x, y)) \\ &\geq S_Y(\eta, y) \to (S_X(\xi, x) \wedge S_Y(\eta, y)) \\ &\geq S_X(\xi, x). \end{aligned}$$

Hence $\varphi(f)$ is continuous. □

By Proposition 4.4, Proposition 4.5 and Proposition 4.9, we have

Theorem 4.10. The category L-**NGCS** is cartesian-closed.

5. Remarks

- Let R be a filter-theoretical stratified L-generalized convergence structure on X. Define $S_R : \mathbb{N}(X) \times X \longrightarrow L$ by $S_R(\xi, x) = R(\mathcal{F}_\xi, x)$. Then by Proposition 2.8, S_R is a net-theoretical L-generalized convergence structure on X. Also, if $f : (X, R_X) \longrightarrow (Y, R_Y)$ is a continuous map between two filter-theoretical L-generalized convergence spaces, then also $f : (X, S_{R_X}) \longrightarrow (Y, S_{R_Y})$ is continuous by Proposition 2.2(4). Thus the category of filter-theoretical L-generalized convergence spaces can be viewed as a subcategory of L-**NGCS**.
- In [6], Jäger defined several subcategories of **S***L***-FCS**, such as stratified L-limit spaces, stratified L-principal convergence spaces, stratified L-topological convergence spaces, et al., and then investigated their topologicalness and cartesian-closedness. It is natural to ask that how can we define and study the L-limit spaces, the L-principal convergence spaces and the L-topological convergence spaces in net-theoretical sense.

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WEI YAO, DEPARTMENT OF MATHEMATICS, HEBEI UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHIJIAZHUANG 050018, P.R. CHINA

E-mail address: `yaowei0516@163.com`